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LETTER TO THE EDITOR

Bernoulli's method for relativistic brachistochrones

Carlos Farina

Instituto de Física, Universidade Federal do Rio de Janeiro, Cidade Universitária, Ilha do Fundão, Rio de Janeiro, CEP: 21.944, Brazil

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Abstract. We obtain the relativistic brachistochrones for both a uniform electric and uniform gravitational field, applying the same method as that used by Bernoulli in 1696 in order to solve the classic brachistochrone problem.

One of the most familiar problems treated in the calculus of variations is the brachistochrone case. As is well known, the classic brachistochrone is the trajectory joining two points P and Q which minimises the total time spent by a particle on its way from P to Q when this particle, initially at rest, moves under the influence of a constant gravitational field. In its movement, the particle is constrained to slide without friction along the curve joining P and Q.

Jean Bernoulli was the first to find the solution for this classic problem analytically, proposed by himself in 1696. He found that the trajectory was a cycloid†.

The relativistic generalisation of this classic problem has been recently presented [2]. In fact, the authors consider a particle falling in both a uniform electric and uniform gravitational field. They discuss not only the curves which minimise the time of flight measured in the laboratory but also those that minimise the proper time.

We will be concerned here with the first case. The purpose of this letter is to re-obtain these brachistochrones exploiting the same ideas as used by Bernoulli to solve the original problem. (For a discussion of the original solution the reader is referred to the book by Polya [3] and for other simple applications one can also see references [4, 5].) Besides its beauty, one of the advantages of using Bernoulli's method is that one obtains the correct results without going into all the techniques of the calculus of variations. This method is based on the fact that in geometrical optics light satisfies Fermat's principle, i.e. it chooses the trajectory which requires the minimal time of travel. All one has to use is the ray equation (or alternatively Snell's law) and the index of refraction $n = c/v$, with v calculated from the energy conservation theorem with suitable potential.

In order to use this method we write the ray equation [6]

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\sigma}\right) = \nabla n \tag{1}$$

where $\boldsymbol{r}$ determines the position of a general point on the trajectory of the light ray and $\mathrm{d}\sigma$ is the differential of the arc length.

† The same problem was also solved by his brother Jacques Bernoulli, Newton, Leibniz and l'Hopital. All of them found the correct result. It is said that Newton solved the problem on the same day he knew about it [1].

The two cases we are going to treat have, of course, plane symmetry. Then $n = c/v$ is a function of one variable only, say x, the vertical distance from a given horizontal plane to the origin 0, taken at the initial point of motion. Thus we can write equation (1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\{n(x)\hat{\tau}\} = \frac{\mathrm{d}n}{\mathrm{d}x}\hat{\imath} \tag{2}$$

where $\hat{\tau} \equiv \mathrm{d}\boldsymbol{r}/\mathrm{d}\sigma$ and $\hat{\imath}$ are unit vectors along the ray direction and vertical direction, respectively. Multiplying both sides of (2) vectorially by $\hat{\imath}$ and integrating them one obtains immediately

$$\sin\theta/v = v_y/v^2 = k/c \tag{3}$$

where k is a dimensionless constant, θ is the angle between $\hat{\imath}$ and $\hat{\tau}$ and we also used the relation $n = c/v$. Let us apply this method in the two following cases.

(i) Uniform electric field. In this case the energy conservation theorem gives [2]

$$v(x) = c\left(1 - \frac{1}{(1+\alpha x)^2}\right)^{1/2} \qquad \alpha = qE/mc^2. \tag{4}$$

From equations (3) and (4) we can write

$$v_y \equiv \frac{\mathrm{d}y}{\mathrm{d}t} = kc\left(1 - \frac{1}{(1+\alpha x)^2}\right) \tag{5}$$

and using the relation

$$v_x^2 = v^2 - v_y^2 \tag{6}$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(c^2 - \frac{c^2}{(1+\alpha x)^2} - v_y^2\right)^{1/2} \tag{7}$$

where we used equation (4). In order to obtain a differential equation for the brachistochrone all we have to do is divide (5) by (7). This gives, after some very simple algebra,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{(kc)^2[(1+\alpha x)^2 - 1]}{(kc)^2 + [c^2 - (kc)^2](1+\alpha x)^2}\right)^{1/2} \tag{8}$$

which, after integration, leads to Goldstein and Bender's solution [2]

$$y = \int_0^x \mathrm{d}x' \left(\frac{k^2(1+\alpha x') - k^2}{(1-k^2)(1+\alpha x')^2 - k^2}\right)^{1/2}. \tag{9}$$

(ii) Uniform gravitational field. In this case the energy conservation theorem gives [2]

$$v(x) = c[1 - \exp(-2\alpha x)]^{1/2}. \tag{10}$$

Using equations (3), (6) and (10), one can follow exactly the same steps as in the previous case to obtain the differential equation for the brachistochrone:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{k^2[1 - \exp(-2\alpha x)]}{(1-k^2) + k^2\exp(-2\alpha x)}\right)^{1/2} \tag{11}$$

which after integration coincides again with Goldstein and Bender's solution [2].

It is interesting to note that Bernoulli's method applies in both relativistic and non-relativistic cases. This was expected and the reason for that can be seen in the following way. In deducing Snell's law $v_1/v_2 = \sin\theta_1/\sin\theta_2$, which can be done by imposing the minimal time of travel for the light ray (Fermat's principle), the velocities can assume all values, including relativistic ones.

We conclude by saying that it is the simplicity of the method just presented which makes it interesting. Many times, the physical concepts become more apparent when one uses simple methods.

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